Powers of Ideals and Fibers of Morphisms

David Eisenbud and Joe Harris [*]

Abstract

Let $X \subset \mathbb{P}^n = \mathbb{P}^n_F$ be a projective scheme over a field F, and let $\phi : X \to Y$ be a finite morphism. Our main result is a formula in terms of global data for the maximum of $\operatorname{reg} \phi^{-1}(y)$, the Castelnuovo-Mumford regularity of the fibers of ϕ over $y \in Y$, where $\phi^{-1}(y)$ is considered as a subscheme of $\mathbb{P}^n$.

From an algebraic point of view, our formula is related to the theorem of Cutkosky-Herzog-Trung [1999] and Kodiyalam [2000] showing that for any homogeneous ideal I in a standard graded algebra S, $\operatorname{reg} I^t$ can be written as $dt + \epsilon$ for some non-negative integers d, ϵ and all large t. In the special case where I contains a power of S_+ and is generated by forms of a single degree, our formula gives an interpretation of ϵ: it is one less than the maximum of $\operatorname{reg} \phi^{-1}(y)$, where ϕ is the morphism associated to I.

These formulas have strong consequences for ideals generated by generic forms.

Introduction

In this note, all schemes will be projective over an arbitrary field F. For any projective scheme $X \subset \mathbb{P}^n$ we write S_X for the homogeneous coordinate ring of X, and I_X for the homogeneous ideal of X. We denote by $\operatorname{reg} X$ the Castelnuovo-Mumford regularity of I_X (if $X = \mathbb{P}^n$ we make the convention that $\operatorname{reg} X = 1$).

If $\phi : X \to Y$ is a finite morphism, then the degree of the fiber $X_y = \phi^{-1}(y)$ is a semicontinuous function of $y \in Y$, and is thus bounded. It follows

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that the Castelnuovo-Mumford regularity of X_y, where X_y considered as a subscheme of $\mathbb{P}^n$, is also bounded. Our main result in the form of Corollary 2.2, gives an algebraic formula for $\max \operatorname{reg}(X_y)$ in terms of global data.

A particularly interesting case occurs when ϕ is a morphism induced by a linear projection $\phi : \mathbb{P}^n \to \mathbb{P}^s$.

Theorem 0.1. *Let $X \subset \mathbb{P}^n$ be a projective scheme with homogeneous coordinate ring S_X, and let $\phi : X \to \mathbb{P}^s$ be a linear projection whose center does not meet X, defined by an $s+1$-dimensional vector space of linear forms V. Let $I \subset S_X$ be the ideal generated by V, and let $\mathfrak{m}$ be the maximal homogeneous ideal of S_X. The maximum of the Castelnuovo-Mumford regularities of the fibers of ϕ over closed points of $\mathbb{P}^s$ is one more than the least ϵ such that, for large t,*

$$\mathfrak{m}^{t+\epsilon} \subset I^t.$$

In the situation of Theorem 0.1 the number $t+\epsilon$ is equal, for large t, to the Castelnuovo-Mumford regularity of I^t or the corresponding ideal sheaf (see §1). Thus Theorem 0.1 clarifies the following beautiful result of Cutkosky-Herzog-Trung [1999], Kodiyalam[2000], and Trung-Wang [2005], at least in a special case.

Theorem 0.2. *If I is a homogeneous ideal in the polynomial ring $S = F[x_0, \ldots, x_n]$, and M is a finitely generated graded module over S, then there are non-negative integers d, ϵ such that*

$$\operatorname{reg}(I^t M) = dt + \epsilon \qquad \textit{for all } t \gg 0. \qquad \square$$

If I is generated by forms of a single degree δ and contains a nonzerodivisor on M, then $d = \delta$. More generally, Kodiyalam [2000] proves that d is the smallest number δ such that $I^t M = I_{\leq \delta} I^{t-1} M$ for large t, where $I_{\leq \delta}$ denotes the ideal generated by the elements of I having degree at most δ.

By contrast, the value of ϵ has been mysterious. Theorem 0.1 gives an interpretation of ϵ in a special case. This seems to be new even for ideals generated in a single degree in a polynomial ring in 2 variables, where Theorem 0.1 yields the following.

Corollary 0.3. *Suppose that $I \subset F[x, y]$ is an ideal generated by a vector space V of forms of degree d, and that F is algebraically closed. Assume that the greatest common divisor of the forms in V is 1. For $V' \subset V$, let $r_{V'}$ be the degree of the greatest common divisor of the forms in V', and let*

$$r := \max\{r_{V'} \mid V' \subset V \textit{ a subspace of codimension } 1\}.$$

If $t \gg 0$, then $\operatorname{reg} I^t = dt + r - 1$. $\qquad\qquad\qquad\qquad\qquad$ □

The corresponding result holds in the case of polynomial rings in more variables, (and also follows from Theorem 0.1) if we assume that I is primary to the maximal homogeneous ideal and redefine $r_{V'}$ to be the maximal degree in which the local cohomology module $H^1_{\mathfrak{m}}(R/(V'R))$ is nonzero (Proposition 1.2).

In the case of 2 variables we may think of V as defining a morphism $\mathbb{P}^1 \to \mathbb{P}(V)$. When this morphism is birational the number r can also be interpreted as the maximum multiplicity of a point on the image curve.

The first author and Roya Beheshti [2008] have conjectured that the regularity of every fiber of a general linear projection of a smooth projective variety X to $\mathbb{P}^{\dim X + c}$, for $c \geq 1$, is bounded by $1 + (\dim X)/c$. Translating this conjecture by means of Theorem 0.1, we get:

Conjecture 0.1 (Beheshti-Eisenbud [2008])**.** *Let R be a standard graded F-algebra of dimension $n + 1$, and let $\mathfrak{m}$ be the maximal homogeneous ideal of R. Suppose that R is a domain with isolated singularity. If F is infinite and $I \subset R$ is an ideal generated by $n + 1 + c$ general linear forms, then*

$$\mathfrak{m}^{t+\epsilon} \subset I^t \quad \text{for all } t \gg 0$$

holds with $\epsilon = \lfloor n/c \rfloor$.

It is easy to see that Conjecture 0.1 holds if $c > n$, and it is known to hold in many other cases as well (see Beheshti-Eisenbud [2008] for a survey), This also gives some new information about ideals generated by generic forms of higher degree. The following is a typical example. Amazingly, we can say no more than this even if we assume that R is the polynomial ring $F[x_0, \ldots, x_n]$.

Corollary 0.4. *Suppose that R is a standard graded algebra of dimension $n + 1$ over a field of characteristic 0, and that R has at most an isolated singularity. If $I = (f_1, \ldots, f_{n+2})$ is an ideal generated $n + 2$ generic forms of degree d, and $n \leq 14$, then*

$$\mathfrak{m}^{t+n} \subset I^t \quad \text{for all } t \gg 0$$

Proof. The linear series given by $f_1, \ldots, f_{n+2}$ defines a generic linear projection of $\operatorname{Proj} R$. By Mather [1971], generic projections in this range of dimensions are stable maps. Mather [1973] gives a local classification of the

multigerms of such stable maps, from which it follows that the degree of the fibers, and thus their regularities, are bounded by $n+1$. The desired formula now follows from Theorem 0.1. $\qquad\square$

We do not currently know any function ϵ of $\dim R$ and c alone that makes the formula in Conjecture 0.1 true. But there is an elementary estimate, whose proof we will give in §1:

Proposition 0.5. *Let R be a standard graded F-algebra of dimension $n+1$, and let $\mathfrak{m}$ be the maximal homogeneous ideal of R. If $I \subset R$ is an ideal generated by linear forms, and if R/I has finite length, then*

$$\mathfrak{m}^{t+\epsilon} \subset I^t \quad \text{for all } t \gg 0$$

holds with $\epsilon = \operatorname{reg} R - 1$. If $X = \operatorname{Proj} R$ is geometrically reduced and connected in codimension 1, then the same formula holds with $\epsilon = \deg X - \operatorname{codim} X$.

It was conjectured by the first author and Shiro Goto in [1984] that if $X = \operatorname{Proj} R$ is geometrically reduced and connected in codimension 1, then $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$, which would say that the first bound given is always sharper than the second, as well as more general.

In Section 1 we prove a sharp form of Theorem 0.2 in the special case of interest for this paper. We also give the proof of a generalization of Proposition 0.5. Section 2 contains our main result, from which we derive Theorem 0.1.

We are grateful to Craig Huneke, with whom we first discussed the problem of identifying the number ϵ in Theorem 0.2. After some experiments using Macaulay2 [M2], he suggested the result in Corollary 0.3, which led us to the results of this paper.

1 The Regularity of Powers

Throughout this paper we write $S = F[x_0, \ldots, x_n]$ and set $\mathfrak{m} = (x_0, \ldots, x_n)$, the homogeneous maximal ideal.

In the case of most interest for this paper, Theorem 0.2 can be strengthened as follows. The result also sharpens Theorem 4 of Chandler [1997], where the line of research leading to Theorem 0.2 began.

Proposition 1.1. *Let M be a finitely generated graded S-module generated in degree 0, and let $I \subset S$ be a homogeneous ideal generated by forms of degree d. If M/IM has finite length, but M does not, then we may write*

1. $\operatorname{reg} M/I^t M = dt + f_t - 1$, *with* $f_1 \geq f_2 \geq \cdots \geq 0$.

2. $\operatorname{reg} I^t M = dt + e_t$, *with* $e_1 \geq e_2 \geq \cdots \geq 0$.

Moreover, $e := \inf\{e_t\} = \inf\{f_t\}$, and we have $\operatorname{reg} I^t M = dt + e$ for $t \gg 0$.

Proof. We first prove the inequalities of part 1. Since $M/I^t M$ has finite length the assertion $\operatorname{reg} M/I^t M = dt + f_t - 1$ means that f_t is the smallest number such that $I^t M$ contains all the graded components of M with degree $\geq dt + f_t + 1$. By our hypotheses on the degrees of generators of M and I, this is equivalent to the assertion $\mathfrak{m}^{f_t+1} I^t M = \mathfrak{m}^{dt+f_t+1} M$. A priori we have $\mathfrak{m}^{f_t+1} I^t M \subset \mathfrak{m}^{d+f_t+1} I^{t-1} M \subset \mathfrak{m}^{dt+f_t+1} M$, so if $\operatorname{reg} M/I^t M \leq dt + f_t$ then these three terms are all equal.

From these equalities we deduce

$$\mathfrak{m}^{f_t+1} I^{t+1} M = I\mathfrak{m}^{f_t+1} I^t M = I\mathfrak{m}^{d+f_t+1} I^{t-1} M = \mathfrak{m}^{d+f_t+1} I^t M = \mathfrak{m}^{d(t+1)+f_t+1} M$$

so $f_{t+1} \leq f_t$. Considering the degrees of the generators of I and M we see that $f_t + 1 \geq 0$ for each t, completing the proof of part 1.

Turning to the assertion of part 2, it is obvious from the consideration of degrees that $e_t \geq 0$. To prove that $e_t \geq e_{t+1}$, let N be the largest submodule of finite length in M. If $N = 0$, then since $M/I^t M$ has finite length, we see from the local cohomology characterization of regularity that

$$\operatorname{reg} I^t M = \max(\operatorname{reg} M, 1 + \operatorname{reg} M/I^t M)$$

so part 2 follows from part 1 in this case. Moreover, since $\operatorname{reg} M/I^t M$ increases without bounds, we see that for large t we will have $e_t = f_t$.

We can reduce the general case to the case $N = 0$ by considering the exact sequence

$$0 \to I^t M \cap N \to I^t M \to I^t(M/N) \to 0.$$

Since $I^t M \cap N$ has finite length, while $I^t(M/N)$ has no finite length submodule except 0,

$$\operatorname{reg} I^t M = \max(\operatorname{reg}(I^t \cap N), \operatorname{reg}(I^t(M/N))).$$

If we replace t by $t + 1$ the term $\mathrm{reg}(I^t \cap N)$ does not increase, while $\mathrm{reg}(I^t(M/N))$ increases by at most d, proving that $e_t \geq e_{t+1}$. Because $\mathrm{reg}(I^t(M/N))$ grows without bound, it eventually dominates, and we again get $e_t = f_t$ for large t. □

We remark that Proposition 1.1 does not hold if we drop the assumption that M/IM has finite length. As shown by Sturmfels [2000] it is not true in general that $\mathrm{reg}\, I^2 M \leq \mathrm{reg}\, IM + d$. For example, with $M = S$ and char $F \neq 2$, the ideal associated to a triangulation of the projective plane has a linear resolution ($\mathrm{reg}\, I = 3$), but its square does not ($\mathrm{reg}\, I^2 = 7 > 2 \times 3$). Conca [2006], gives examples with $\mathrm{reg}\, I^n = n \, \mathrm{reg}\, I$ but $\mathrm{reg}\, I^{n+1} > (n+1)\, \mathrm{reg}\, I$ for arbitrary n.

We now turn to the proof of Proposition 0.5. The first estimate is a Corollary of the following result:

Proposition 1.2. *Let M be a graded module of dimension n over a polynomial ring $S = F[x_0, \ldots, x_r]$, and let I be an ideal generated by forms of degree d such that M/IM has finite length. For every $t > 0$ we have*

$$\mathrm{reg}\, M/I^t M \leq td + \mathrm{reg}\, M + (n-1)(d-1) - 1 \qquad \text{for every } t > 0.$$

Moreover, equality holds when the generators of I form a regular sequence on M.

Proof. If I is generated by a regular sequence on M, then one can obtain a resolution of $M/I^t M$ by tensoring a resolution of M with one for S/I^t (obtained, for example, as an Eagon-Northcott complex) and from this one computes the regularity at once. (This much does not use the hypothesis that M/IM has finite length.)

When M/IM has finite length, we may begin by replacing I by a smaller ideal, generated by a system of parameters of degree d on M—in this case, the regularity of $M/I^t M$ is simply the degree of the socle, so it can only increase. It is not hard to give an elementary argument using induction on t. Alternately, the result of Caviglia [2007] (see also Sidman [2002]) shows that $\mathrm{reg}\, M/I^t M = \mathrm{reg}(M \otimes S/I^t) \leq \mathrm{reg}\, M + \mathrm{reg}\, S/I^t = \mathrm{reg}\, M + (t-1)d + (d-1)\dim M$ where the last equality follows from the argument above and the fact that I is generated by a regular sequence on S. □

Proof of Proposition 0.5. For the first estimate we set $d = 1$ in Proposition 1.2 and use the fact that the regularity of R/I^t is the largest number s such

that $\mathfrak{m}^s \not\subset I^t$. For the second estimate we first observe that it suffices to do the case where the number of linear forms is $\dim X$ – that is, a fiber of the projection is just the intersection of X with a plane of complementary dimension. Under the hypotheses given, such a plane section of X is a scheme of degree $\deg X$ and is nondegenerate. The latter condition implies that the regularity of the fiber is bounded above by $\deg X - \operatorname{codim} X + 1$. Theorem 2.1 now gives the desired equality. $\qquad\square$

2   The Fibers of Finite Morphisms

We now turn to the result that will allow us to give the maximum regularity of the fibers of a finite morphism in terms of global data (Corollary 2.2).

Theorem 2.1. *Let X be a scheme, and let $\phi : X \to \mathbb{P}^s$ be a finite morphism, corresponding to the line bundle $\mathcal{L} = \phi^* \mathcal{O}_{\mathbb{P}^s}(1)$ and the space of global sections $V = \phi^*(H^0 \mathcal{O}_{\mathbb{P}^s}(1)) \subset H^0 \mathcal{L}$. Let M be a coherent sheaf on X, and let $W \subset H^0(M)$ be a space of sections. The following are equivalent:*

1. For every integer $t \gg 0$, the map

$$\operatorname{Sym}_t(V) \otimes W \to H^0(\mathcal{L}^t \otimes M)$$

is surjective.

2. For every closed point $p \in \mathbb{P}^s$, the restriction map

$$W \to H^0(\phi^{-1}(p) \otimes M)$$

is surjective.

3. The map of sheaves
$$\mu : W \otimes \mathcal{O}_{\mathbb{P}^s} \to \phi_* M.$$
is surjective

Proof. $1 \Leftrightarrow 3$: By Serre's Vanishing Theorem, the surjectivity of μ is equivalent to the surjectivity, for $t \gg 0$, of the map

$$W \otimes \operatorname{Sym}_t(V) = W \otimes H^0(\mathcal{O}_{\mathbb{P}(V)}(t)) \to H^0(\phi_*(M)(t)).$$

Thus

$$\phi_*(M)(t) = \phi_*(M) \otimes_{\mathcal{O}_{\mathbb{P}^s}} \mathcal{O}_{\mathbb{P}^s}(t) = \phi_*(M \otimes_{\mathcal{O}_X} \phi^*\mathcal{O}_{\mathbb{P}^s}(t)) = \phi_*(M \otimes_{\mathcal{O}_X} \mathcal{L}^t).$$

Taking global sections, this gives

$$H^0(\phi_*(M)(t)) = H^0(\phi_*(M \otimes_{\mathcal{O}_X} \mathcal{L}^t)) = H^0(M \otimes_{\mathcal{O}_X} \mathcal{L}^t),$$

proving that assertion 1 is equivalent to assertion 3.

$2 \Leftrightarrow 3$: Since ϕ_*M is coherent, the surjectivity of μ is equivalent by Nakayama's Lemma to the the surjectivity of all the restriction maps $W = W \otimes \mathcal{O}_{\{p\}} \to (\phi_*M) \otimes \mathcal{O}_{\{p\}}$, where p runs over the closed points of $\mathbb{P}^s$ (or just of the image of X). Using the finiteness of ϕ, we can make the identifications

$$
\begin{aligned}
(\phi_*M) \otimes \mathcal{O}_{\{p\}} \quad &= H^0((\phi_*M) \otimes \mathcal{O}_{\{p\}}) \\
&= H^0(\phi_*(M \otimes \phi^*\mathcal{O}_{\{p\}})) \\
&= H^0((M \otimes \phi^*\mathcal{O}_{\{p\}})) \\
&= H^0(M \otimes \mathcal{O}_{\phi^{-1}(p)}),
\end{aligned}
$$

so assertion 2 is also equivalent to the surjectivity of μ. $\qquad\square$

Corollary 2.2. *Suppose that $X \subset \mathbb{P}^n$ is a projective scheme, and $\phi : X \to \mathbb{P}^s$ is a finite morphism, corresponding to a linear system $V \subset H^0(\mathcal{L})$. The maximum regularity of a fiber of ϕ over a closed point of $\mathbb{P}^s$ is one more than the minimum integer ϵ such that*

$$H^0(\mathcal{O}_{\mathbb{P}^n}(\epsilon)) \otimes \mathrm{Sym}_t(V) \to H^0(\mathcal{L}^t(\epsilon))$$

is surjective for $t \gg 0$.

Proof. The regularity of a fiber $\phi^{-1}(p)$ is the smallest integer t such that $H^i(\mathcal{I}_{\phi^{-1}(p)}(t-i)) = 0$ for all $i > 0$. For a non-empty fiber $Z = \phi^{-1}(p)$ of dimension 0, only $i = 1$ can be of significance, and the regularity of Z is one more than the minimum ϵ such that $H^1(\mathcal{I}_{\phi^{-1}(p)})(\epsilon) = 0$. Identifying $\mathcal{O}_Z$ with $\mathcal{O}_Z(d)$, the long exact sequence in cohomology shows that this is the least ϵ such that the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n}(\epsilon)) \to H^0(\mathcal{O}_Z(\epsilon)) \cong H^0(\mathcal{O}_Z)$$

is surjective. The Corollary thus follows from the equivalence $1 \Leftrightarrow 2$ of Theorem 2.1 if we take $M = \mathcal{O}_{\mathbb{P}^n}(\epsilon)$ and $W = H^0(\mathcal{O}_{\mathbb{P}^n}(\epsilon))$. $\qquad\square$

Proof of Theorem 0.1. In Corollary 2.2 take $\mathcal{L} = \mathcal{O}_X(1)$, $M = \mathcal{O}_X(e)$, and $W = H^0(M)$. The projection ϕ is finite since the projection center does not meet X. $\qquad\square$

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Author Addresses:

David Eisenbud
Department of Mathematics, University of California, Berkeley, Berkeley CA 94720
eisenbud@math.berkeley.edu

Joe Harris
Department of Mathematics, Harvard University, Cambridge MA 02138
harris@math.harvard.edu